18.05 Lecture 37
May 17, 2005

Final Exam Review - solutions to practice final.

1. $f(x|v) = \{uv^{-u}x^{u-1}e^{-(\frac{x}{v})^u}$ for $x \geq 0; 0$ otherwise.$\}$
Find the MLE of v.
$\rightarrow$ Maximize v in the likelihood function = joint p.d.f.

$$\psi(v) = u^n v^{-nu} (\prod x_i)^{u-1} e^{-\sum (x_i/v)^u}$$

$\log \psi(v) = n \log u - nu \log v + (u-1)\log(\prod x_i) - \sum (\frac{x_i}{v})^u$
Maximize with respect to v.

$$\frac{\partial(\log \psi(v))}{\partial v} = \frac{-nu}{v} - \sum x_i^u \frac{\partial}{\partial v} v^{-u} = \frac{-nu}{v} + u \sum x_i^u v^{-(u+1)} = 0$$

$$n = \frac{v}{u}\left(\frac{u\sum x_i^u}{v^{u+1}}\right) = \frac{\sum x_i^u}{v^u}$$

$$v^u = \frac{1}{n}\sum x_i^u$$

$$v = \left(\frac{1}{n}\sum x_i^u\right)^{\frac{1}{u}} \rightarrow \textbf{MLE}$$

2. $X_i, ..., X_n \sim U[0, \theta]$
$f(x|\theta) = \frac{1}{\theta} I(0 \leq x \leq \theta)$
Prior:

$$f(\theta) = \frac{192}{\theta^4} I(\theta \geq 4)$$

Data: $X_1 = 5, X_2 = 3, X_3 = 8$
Posterior: $f(\theta|x_1, ..., x_n) \sim f(x_1, ..., x_n|\theta) f(\theta)$
$f(x_1, ..., x_n|\theta) = \frac{1}{\theta^n} I(0 \leq \text{ all x's } \leq \theta) = \frac{1}{\theta^n} I(\max(X_1, ..., X_n) \leq \theta)$
$f(\theta|x_1, ..., x_n) \sim \frac{1}{\theta^{n+4}} I(\theta \geq 4) I(\max(x_1, ..., x_n) \leq \theta) \sim \frac{1}{\theta^{n+4}} I(\theta \geq 8)$

Find constant so it integrates to 1.

$$1 = \int_8^\infty \frac{1}{\theta^{n+4}} d\theta \; \overrightarrow{n = 3} \; 1 = \int_8^\infty c\theta^{-7} d\theta \rightarrow \frac{c\theta^{-6}}{-6}\Big|_8^\infty = \frac{c}{6} 8^{-6} = 1$$

$$c = 6 \times 8^6$$

3. Two observations (X_1, X_2) from f(x)
$H_1 : f(x) = 1/2, I(0 \leq x \leq 2)$
$H_2 : f(x) = \{1/2, 0 \leq x \leq 1, 2/3, 1 < x \leq 2\}$
$H_3 : f(x) = \{3/4, 0 \leq x \leq 1, 1/4, 1 < x \leq 2\}$
δ minimizes $\alpha_1(\delta) + 2\alpha_2(\delta) + 2\alpha_3(\delta)$
$\alpha_i(\delta) = \mathbb{P}(\delta \neq H_i | H_i)$
Find $\sum \xi(i)\alpha_i$, Decision rule picks $\xi(i) f_i(x_1, ..., x_n) \rightarrow$ max for each region.

$\xi(i)f_i(x_1)f_i(x_2)$	H_1	H_2	H_3
both $x_1, x_2 \in [0,1]$	$(1)(1/2)(1/2) = 1/4$	$(2)(1/3)(1/3) = 1/3$	$(2)(3/4)(3/4) = \mathbf{9/8}$
point in $[0,1], [1,2]$	$(1)(1/2)(1/2) = 1/4$	$(2)(1/3)(2/3) = \mathbf{4/9}$	$(2)(3/4)(1/4) = 3/8$
both in $[1,2]$	$(1)(1/2)(1/2) = 1/4$	$(2)(1/3)(2/3) = \mathbf{8/9}$	$(2)(1/4)(1/4) = 1/8$

Decision Rule:
$\delta = \{H_1 : \text{ never pick }, H_2 : \text{ both in [1, 2], one in [0, 1], [1, 2] }, H_3 : \text{ both in [0, 1]}\}$
If two hypotheses:

$$\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} > \frac{\xi(2)}{\xi(1)}$$

Choose $H_1, \leq H_2$

4.

$$f(x|\mu) = \{\frac{1}{x\sqrt{2\pi}} e^{\frac{-1}{2}(\ln x - \mu)^2} \text{ for } x \geq 0, 0 \text{ for } x < 0\}$$

If X has this distribution, find distribution of $\ln X$.
$Y = \ln X$
c.d.f. of Y: $\mathbb{P}(Y \leq y) = (\ln x \leq y) = \mathbb{P}(x \leq e^y) = \int_0^{e^y} f(x)dx$
However, you don't need to integrate.
p.d.f. of Y, $f(y) = \frac{\partial}{\partial y}\mathbb{P}(Y \leq y) = f(e^y) \times e^y$

$$= \frac{1}{e^y\sqrt{2\pi}} e^{\frac{-1}{2}(\ln e^y - \mu)^2} \times e^y = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(y-\mu)^2} \sim N(\mu, 1)$$

5. n = 10, $H_1 : \mu = -1, H_2 : \mu = 1$
$\alpha = 0.05 \rightarrow \alpha_1(\delta) = \mathbb{P}_1(\frac{f_1}{f_2} < c)$

$$\delta = \{H_1 : \frac{f_1(\overrightarrow{x})}{f_2(\overrightarrow{x})} \geq c, H_2 \text{ if less than}\}$$

$$f_1(\overrightarrow{x}) = \frac{1}{\prod x_i(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum(\ln x_i + 1)^2}$$

$$f_2(\overrightarrow{x}) = \frac{1}{\prod x_i(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum(\ln x_i - 1)^2}$$

$$\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} = e^{-2\sum \ln x_i} \geq c \leftrightarrow \sum \ln x_i \leq c'$$

$$\delta = \{H_1 : \sum \ln x_i \leq c = -4.81, H_2 : \sum \ln x_i > c = -4.81\}$$

$0.05 = \mathbb{P}_1(\sum \ln x_i \geq c) = \mathbb{P}_1(\sum N(-1,1) \geq c) = \mathbb{P}_1(\frac{\sum x_i - n\mu}{\sqrt{n}} \geq \frac{c - n\mu}{\sqrt{n}\sigma}) = \mathbb{P}_1(Z \geq \frac{c-n\mu}{\sqrt{n}\sigma})$

$$\frac{c - n\mu}{\sqrt{n}\sigma} = 1.64, c = -4.81$$

Power = 1 - type 2 error $= 1 - \mathbb{P}_2(\delta \neq H_2) = 1 - \mathbb{P}_2(\sum \ln x_i < c) = 1 - \mathbb{P}_2(\sum N(1,1) < c)$

$$= 1 - \mathbb{P}_2(\frac{\sum x_i - n(1)}{\sqrt{n}} < \frac{-4.81 - 10}{\sqrt{10}}) \approx 1$$

6. $H_1 : p_1 = \frac{\theta}{2}, p_2 = \frac{\theta}{3}, p_3 = 1 - \frac{5\theta}{6}, \theta \in [0,1]$
Step 1) Find MLE θ^*
Step 2) $p_1^* = \frac{\theta^*}{2}, p_2^* = \frac{\theta^*}{3}, p_3^* = 1 - \frac{5\theta^*}{6}$
Step 3) Calculate T statistic.

$$T = \sum_{i=1}^{r} \frac{(N_i - np_i^*)^2}{np_i} \sim \chi^2_{r-s-1=3-1-1=1}$$

$$\psi(\theta) = (\frac{\theta}{2})^{N_1}(\frac{\theta}{3})^{N_2}(1 - \frac{5\theta}{6})^{N_3}$$

$$\log\psi(\theta) = (N_1 + N_2)\log(\theta) + N_3\log(1 - \frac{5\theta}{6}) - N_1\log(2)N_2\log(3) \to \ \max\theta$$

$$\frac{\partial}{\partial\theta} = \frac{N_1 + N_2}{\theta} + N_3\frac{-5/6}{1 - \frac{5\theta}{6}}$$

$$N_1 + N_2 - \frac{5}{6}(N_1 + N_2)\theta - \frac{5}{6}N_3\theta = 0$$

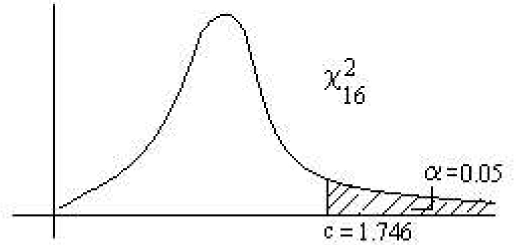
$$\text{solve for } \theta \to \theta^* = \frac{6}{5}(\frac{N_1 + N_2}{n}) = \frac{23}{25}$$

Compute statistic, T = 0.586
$\delta = \{H_1 : T \leq 3.841, H_2 : T > 3.841\}$
Accept H_1

7. $n = 17, \overline{x} = 3.2, x = 0.09$
From $N(\mu, \sigma^2)$
$H_1 : \mu \leq 3$
$H_2 : \mu > 3$
at $\alpha = 0.05$

$$\mathcal{T} = \frac{\overline{x} - \mu_0}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}} \sim t_{n-1} = \frac{3.2 - 3}{\sqrt{\frac{1}{16}(0.09)}} = 2.67$$

Choose decision rule from the chi-square table with 17-1 degrees of freedom:



$\delta : \{H_1 : T < 1.746, H_2 : T > 1.746\}$
H_1 is rejected.

8. Calculate t statistic:

$$T = \sum_{i,j} \frac{(N_{ij} - \frac{N_{+j}N_{i+}}{n})^2}{\frac{N_{+j}N_{i+}}{n}} = 12.1$$

$\chi^2_{(a-1)(b-1)} = \chi^2_{3\times 2} = \chi^2_6$ at 0.05, c = 12.59
$\delta : \{H_1 : T \le 12.59, H_2 : T > 12.59\}$
Accept H_1. But note if confidence level changes, $\alpha \to 0.10, c = 10.64$, would reject H_1

9.
$f(x) = 1/2, I(0 \le x \le 2)$
$F(x) = \int_{-\infty}^{x} f(t)dt = x/2, x \le 2$

x:	0.02	0.18	0.20	...
F(x):	0.01	0.09	0.10	...
F(x) before:	0	0.1	0.2	...
F(x) after:	0.1	0.2	0.3	...
diff F(x) before:	0.01	0.01	0.1	...
diff F(x) after:	0.09	0.11	0.2	...

$\Delta_n = |F(x) - F_n(x)|$
max $\Delta_n = 0.295$
c for $\alpha = 0.05$ is 1.35
$D_n = \sqrt{10}(0.295) = 0.932872$
$\delta = \{H_1 : 0.932872 \le 1.35, H_2 : 0.932872 > 1.35\}$
Accept H_1

** End of Lecture 37

*** End of 18.05 Spring 2005 Lecture Notes.